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A covariant action with a constraint and Feynman rules for fermions in open superstring field theory

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Abstract: In a way analogous to type-IIB supergravity, we give a covariant action for the fermion field supplemented with a constraint which should be imposed on equations of motion, in Berkovits' open superstring field theory. From this action we construct Feynman rules for computing perturbative amplitudes for fermions. We show that on-shell tree level 4-point amplitudes computed by using these rules coincide with those of the first quantization formalism.

Keywords: Superstrings and Heterotic Strings, D-branes, String Field Theory.

Contents

1. Introduction

In superstring theory supersymmetry plays crucial roles. Therefore it is very important to consider the fermion sector as well as the boson sector. In this paper we consider this issue in the open superstring field theory proposed by Berkovits [1]. In [1] only the bosonic part was given, and some attempts to introduce fermions have been made in [2], where covariant equations of motion for the fermion field have been given, and unfortunately it is impossible to write down an action from which those equations are derived. Some noncovariant actions have also been given in [2]. This has been achieved by introducing two or more additional string fields, but those string fields consist of both bosons and fermions, and the apparent forms of the bosonic parts of those actions are different from that of [1].

This situation is reminiscent of ordinary field theories with self-dual forms: we can easily write down field equations, but naive attempts to construct covariant actions from which those equations are derived fail. [3] (However introduction of an auxiliary field leads to a successful formulation. [4]) One of this kind of theory is type-IIB supergravity, which has a 4-form with self-dual 5-form field strength. What we usually do in this theory is as follows: we write down an action pretending that the 4-form has both the self-dual and the anti self-dual part, and after we compute equations of motion for the self-dual field and others we impose the self-duality condition on them. In this paper we will apply the same procedure in the open superstring field theory. i.e. we introduce an additional string field corresponding to the anti self-dual part, write down a covariant action, and impose a constraint. We will show that the equations of motion derived from our action reduce to those of [2] under the constraint. Then we will use the action for deriving Feynman rules for computing perturbative amplitudes for fermions. We show that 4-point on-shell tree level amplitudes with fermions computed according to these rules coincide with those of the first quantization formalism.

2. A covariant action for fermions with a constraint

Let us recall type-IIB supergravity. This theory has a 4-form, and its field strength is self-dual. In general the kinetic term of a $(p-1)$-form is given by $F_{\mu_1\mu_2\ldots\mu_p}F^{\mu_1\mu_2\ldots\mu_p}$, and when p is half of spacetime dimension D and $D = 2 \bmod 4$, this is equal to $2F^{+}_{\mu_1\mu_2\ldots\mu_p}F^{-\,\mu_1\mu_2\ldots\mu_p}$, where $F^{\pm}_{\mu_1\mu_2\ldots\mu_p}$ are the self-dual and the anti self-dual part of the field strength respectively. Since both parts appear in this form of kinetic term, it does not extend to the case with only the self-dual part.

We usually detour around this problem by writing down an action assuming temporarily that the 5-form field strength has both self-dual and anti self-dual part, and imposing additional self-duality constraint after deriving equations of motion. The action for the metric, dilaton, NSNS B-field, and RR forms C_n is

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\bigg[& e^{-2\phi}\left(R + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}|H_3|^2\right) - \\ & - \frac{1}{2}\left(|F_1|^2 + |\widetilde{F}_3|^2 + \frac{1}{2}|\widetilde{F}_5|^2\right)\bigg] - \\ -\frac{1}{4\kappa^2}\int C_4\wedge H_3\wedge F_3\,, \end{aligned} \tag{2.1}$$

where H and F_{n+1} are field strengths of B and C_n respectively, and

$$\begin{aligned} \widetilde{F}_3 &= F_3 - C_0\wedge H_3\,, \\ \widetilde{F}_5 &= F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3\,. \end{aligned} \tag{2.2}$$

The self-duality condition is $*\widetilde{F}_5 = \widetilde{F}_5$. This is imposed on the solutions, and not on the action. The equation of motion for C_4 derived from the above action is $d*\widetilde{F}_5 = H_3\wedge F_3$ and this is reduced to the Bianchi identity under the self-duality condition.

Next we consider the fermion sector of Berkovits' open superstring field theory [1] corresponding to one single BPS D-brane. Extension to non-BPS D-branes or multiple D-brane case is straightforward. A natural string field Ψ for fermions has $n_p = 1/2$ and $n_g = 0$, where n_p and n_g are picture number and ghost number respectively. (For the assignment of these numbers see, for instance, [5].) This field corresponds to ϕ-charge $-1/2$ vertex operators in the first quantization formalism. At the linearized level this field should have the following gauge symmetry,

$$\delta\Psi = Q_B\Lambda_{1/2} + \eta_0\Lambda_{3/2}\,, \tag{2.3}$$

where Λ_n are parameters with $(n_p, n_g) = (n, -1)$, and the equation of motion should be $Q_B\eta_0\Psi = 0$.

In the oscillator expression this field is expanded by the following states, constructed by acting indicated oscillators so that they have indicated n_g and Grassmann parity:

$$\xi_0\Big\{\beta_{n\leq-1}, \gamma_{n\leq0}, b_{n\leq-1}, c_{n\leq0}, L^m_{n\leq-1}, G^m_{n\leq-1}; n_g = 0, \text{Grassmann even}\Big\}\,\Big|\Omega^A_{-1/2}\Big\rangle\,,$$

level ($L_0 - \alpha' k^2$)	states with ξ_0 and without c_0	states with ξ_0 and with c_0	states without ξ_0 and without c_0	states without ξ_0 and with c_0
0	$\xi_0 \left\vert \Omega^A_{-1/2}, k \right\rangle$	none	$b_{-1} \left\vert \widetilde{\Omega}^A_{1/2}, k \right\rangle$	none
1	$\xi_0 \beta_{-1} \gamma_0 \left\vert \Omega^A_{-1/2}, k \right\rangle$ $\xi_0 b_{-1} \gamma_0 \left\vert \widetilde{\Omega}^A_{-1/2}, k \right\rangle$ $\xi_0 \alpha^\mu_{-1} \left\vert \Omega^A_{-1/2}, k \right\rangle$ $\xi_0 \psi^\mu_{-1} \left\vert \widetilde{\Omega}^A_{-1/2}, k \right\rangle$	$\xi_0 c_0 \beta_{-1} \left\vert \widetilde{\Omega}^A_{-1/2}, k \right\rangle$ $\xi_0 c_0 b_{-1} \left\vert \Omega^A_{-1/2}, k \right\rangle$	$b_{-2} \left\vert \widetilde{\Omega}^A_{1/2}, k \right\rangle$ $\beta_{-2} \left\vert \Omega^A_{1/2}, k \right\rangle$ $b_{-1} \alpha^\mu_{-1} \left\vert \widetilde{\Omega}^A_{1/2}, k \right\rangle$ $b_{-1} \psi^\mu_{-1} \left\vert \Omega^A_{1/2}, k \right\rangle$ $b_{-2} b_{-1} \gamma_1 \left\vert \Omega^A_{1/2}, k \right\rangle$ $\beta_{-2} b_{-1} \gamma_1 \left\vert \widetilde{\Omega}^A_{1/2}, k \right\rangle$	none

Table 1: Low-lying states for the expansion of Ψ in the flat background.

$$
\begin{aligned}
&\xi_0 \Big\{ \beta_{n\le -1}, \gamma_{n\le 0}, b_{n\le -1}, c_{n\le 0}, L^m_{n\le -1}, G^m_{n\le -1}; n_g = 0, \text{Grassmann odd} \Big\} \left\vert \widetilde{\Omega}^A_{-1/2} \right\rangle , \\
&\Big\{ \beta_{n\le -2}, \gamma_{n\le 1}, b_{n\le -1}, c_{n\le 0}, L^m_{n\le -1}, G^m_{n\le -1}; n_g = -1, \text{Grassmann even} \Big\} \left\vert \Omega^A_{1/2} \right\rangle , \\
&\Big\{ \beta_{n\le -2}, \gamma_{n\le 1}, b_{n\le -1}, c_{n\le 0}, L^m_{n\le -1}, G^m_{n\le -1}; n_g = -1, \text{Grassmann odd} \Big\} \left\vert \widetilde{\Omega}^A_{1/2} \right\rangle , \qquad (2.4)
\end{aligned}
$$

where $\left\vert \Omega^A_n \right\rangle = c(0) e^{n\phi(0)} \Sigma^A(0) \left\vert 0 \right\rangle$ and $\vert \widetilde{\Omega}^A_n \rangle = c(0) e^{n\phi(0)} \widetilde{\Sigma}^A(0) \left\vert 0 \right\rangle$. Σ^A and $\widetilde{\Sigma}^A$ are spin operators with positive and negative chirality respectively. Note that on $\vert \Omega^A_{\pm 1/2} \rangle$ and $\vert \widetilde{\Omega}^A_{\pm 1/2} \rangle$, β, γ and G^m have integer mode numbers. The operators $e^{-\frac{1}{2}\phi} \Sigma^A$ and $e^{-\frac{1}{2}\phi} \widetilde{\Sigma}^A$ should be regarded as Grassmann odd and even respectively, then all the above states are Grassmann odd. We set the coefficients of these states Grassmann odd, so that they represent fermions. Then Ψ is Grassmann even.

Table 1 shows low lying states for the expansion of Ψ in the flat background. In the table, $\left\vert \Omega^A_n, k \right\rangle = c(0) e^{n\phi(0)} \Sigma^A(0) e^{ik\cdot X(0)} \left\vert 0 \right\rangle$ and $\left\vert \widetilde{\Omega}^A_n, k \right\rangle = c(0) e^{n\phi(0)} \widetilde{\Sigma}^A(0) e^{ik\cdot X(0)} \left\vert 0 \right\rangle$.

Naively kinetic term of Ψ is $\langle\langle (Q_B \Psi)(\eta_0 \Psi) \rangle\rangle$, but this vanishes because of the picture number conservation law. One may think we can introduce picture changing operators to give correct kinetic term, but it is well known that this causes divergent contact term problems [6], and modifies the equation of motion.

Thus it seems impossible to construct a consistent kinetic term for Ψ. However, as has been done in [2], we can construct a nonlinear extension of equations of motion:

$$\eta_0 (G^{-1} (Q_B G)) = -(\eta_0 \Psi)^2 \,, \qquad (2.5)$$

$$Q_B (G (\eta_0 \Psi) G^{-1}) = 0 \,, \qquad (2.6)$$

and gauge symmetry:

$$\delta G = G (\eta_0 \Lambda_1 - \{\eta_0 \Psi, \Lambda_{1/2}\}) + (Q_B \Lambda_0) G \,, \qquad (2.7)$$

$$\delta \Psi = \eta_0 \Lambda_{3/2} + [\Psi, \eta_0 \Lambda_1] + Q_B \Lambda_{1/2} + \{G^{-1}(Q_B G), \Lambda_{1/2}\} \,, \qquad (2.8)$$

where $G = e^\Phi$, and Φ is the string field for bosons.

level $(L_0-\alpha' k^2)$	states with ξ_0 and without c_0	states with ξ_0 and with c_0	states without ξ_0 and without c_0	states without ξ_0 and with c_0
0	$\xi_0 \left\|\widetilde{\Omega}^A_{-3/2},k\right\rangle$	$\xi_0 c_0 \beta_0 \left\|\Omega^A_{-3/2},k\right\rangle$	none	none
1	$\xi_0\beta_0\gamma_{-1} \left\|\widetilde{\Omega}^A_{-3/2},k\right\rangle$ $\xi_0\beta_0 c_{-1} \left\|\Omega^A_{-3/2},k\right\rangle$ $\xi_0\alpha^\mu_{-1} \left\|\widetilde{\Omega}^A_{-3/2},k\right\rangle$ $\xi_0\psi^\mu_{-1} \left\|\Omega^A_{-3/2},k\right\rangle$	$\xi_0 c_0\beta_{-1} \left\|\Omega^A_{-3/2},k\right\rangle$ $\xi_0 c_0 b_{-1} \left\|\widetilde{\Omega}^A_{-3/2},k\right\rangle$ $\xi_0 c_0\beta_0\alpha^\mu_{-1} \left\|\Omega^A_{-3/2},k\right\rangle$ $\xi_0 c_0\beta_0\psi^\mu_{-1} \left\|\widetilde{\Omega}^A_{-3/2},k\right\rangle$ $\xi_0 c_0(\beta_0)^2\gamma_{-1} \left\|\Omega^A_{-3/2},k\right\rangle$ $\xi_0 c_0(\beta_0)^2 c_{-1} \left\|\widetilde{\Omega}^A_{-3/2},k\right\rangle$	$\beta_{-1} \left\|\widetilde{\Omega}^A_{-1/2},k\right\rangle$ $b_{-1} \left\|\Omega^A_{-1/2},k\right\rangle$	none

Table 2: Low-lying states for the expansion of Ξ in the flat background.

Comparing this with ordinary field theories with self-dual forms, we notice that we are in a similar situation: We have equations of motion, but cannot write down an covariant action, in particular kinetic term, which reproduces them. Then it is natural to think about doing the same thing as in type-IIB supergravity. i.e. adding an additional field corresponding to the anti self-dual part, writing down an action, and impose a constraint corresponding to the self-duality condition. Let us call the additional string field Ξ, and we infer the action at the linearized level is given by the product of Ξ and Ψ just as the kinetic terms of forms are given by the product of the self-dual and the anti self-dual part:

$$S_F = -\frac{1}{2g^2}\,\langle\langle (Q_B\Xi)(\eta_0\Psi)\rangle\rangle\ . \tag{2.9}$$

From this we can see that Ξ has $(n_p,n_g) = (-1/2,0)$, and is Grassmann even. In the oscillator expression Ξ is expanded by the following states:

$$\begin{aligned}
&\xi_0\Big\{\beta_{n\le 0},\gamma_{n\le -1},b_{n\le -1},c_{n\le 0},L^m_{n\le -1},G^m_{n\le -1}; n_g=0,\text{Grassmann odd}\Big\}\Big|\Omega^A_{-3/2}\Big\rangle\,,\\
&\xi_0\Big\{\beta_{n\le 0},\gamma_{n\le -1},b_{n\le -1},c_{n\le 0},L^m_{n\le -1},G^m_{n\le -1}; n_g=0,\text{Grassmann even}\Big\}\Big|\widetilde{\Omega}^A_{-3/2}\Big\rangle\,,\\
&\Big\{\beta_{n\le -1},\gamma_{n\le 0},b_{n\le -1},c_{n\le 0},L^m_{n\le -1},G^m_{n\le -1}; n_g=-1,\text{Grassmann odd}\Big\}\Big|\Omega^A_{-1/2}\Big\rangle\,,\\
&\Big\{\beta_{n\le -1},\gamma_{n\le 0},b_{n\le -1},c_{n\le 0},L^m_{n\le -1},G^m_{n\le -1}; n_g=-1,\text{Grassmann even}\Big\}\Big|\widetilde{\Omega}^A_{-1/2}\Big\rangle\,.
\end{aligned} \tag{2.10}$$

Table 2 shows low lying states for the expansion of Ξ in the flat background.

The equations of motion for Ψ and Ξ are

$$Q_B\eta_0\Xi = 0\,, \tag{2.11}$$

$$Q_B\eta_0\Psi = 0\,. \tag{2.12}$$

We have to put the following constraint to eliminate the superfluous degrees of freedom and to make the equation of motion of Ψ or Ξ trivial.

$$Q_B\Xi = \eta_0\Psi\,. \tag{2.13}$$

This condition means that the "self-dual" and "anti self-dual" part correspond to $\frac{1}{2}(Q_B\Xi\pm\eta_0\Psi)$, rather than Ψ and Ξ.

The above action is easily extended to a nonlinear interacting system:

$$S = S_B + S_F \,, \tag{2.14}$$

$$S_B = \frac{1}{2g^2}\left\langle\!\!\left\langle G^{-1}(Q_B G)G^{-1}(\eta_0 G) - \int_0^1 dt G_t^{-1}(\partial_t G_t)\{G_t^{-1}(Q_B G_t), G_t^{-1}(\eta_0 G_t)\}\right\rangle\!\!\right\rangle , \tag{2.15}$$

$$S_F = -\frac{1}{2g^2}\langle\langle (Q_B\Xi)G(\eta_0\Psi)G^{-1}\rangle\rangle \,, \tag{2.16}$$

where $G_t = e^{t\Phi}$. S_B is the bosonic part given in [1].

The equations of motion for Φ, Ψ and Ξ are

$$\eta_0(G^{-1}(Q_B G)) = -\frac{1}{2}(\eta_0\Psi)G^{-1}(Q_B\Xi)G - \frac{1}{2}G^{-1}(Q_B\Xi)G(\eta_0\Psi) \,, \tag{2.17}$$

$$\eta_0(G^{-1}(Q_B\Xi)G) = 0 \,, \tag{2.18}$$

$$Q_B(G(\eta_0\Psi)G^{-1}) = 0 \,. \tag{2.19}$$

The constraint is extended to

$$Q_B\Xi = G(\eta_0\Psi)G^{-1} \,. \tag{2.20}$$

Under this constraint either of the equations of motion for Ψ and Ξ can be regarded as trivial, and the three equations of motion are reduced to eq. (2.5) and eq. (2.6).

The action (2.14) has the following gauge symmetry.

$$\delta G = G(\eta_0\Lambda_1) + (Q_B\Lambda_0)G \,, \tag{2.21}$$

$$\delta\Psi = \eta_0\Lambda_{3/2} + [\Psi, \eta_0\Lambda_1] \,, \tag{2.22}$$

$$\delta\Xi = Q_B\Lambda_{-1/2} + [Q_B\Lambda_0, \Xi] \,. \tag{2.23}$$

This symmetry is consistent with the constraint:

$$\delta(Q_B\Xi - G(\eta_0\Psi)G^{-1}) = [Q_B\Lambda_0, Q_B\Xi - G(\eta_0\Psi)G^{-1}] \,. \tag{2.24}$$

However eq. (2.5) and eq. (2.6) have larger gauge symmetry: the transformations of G and Ψ have an extra parameter $\Lambda_{1/2}$. The action (2.14) does not have this symmetry. Thus we have an enhanced symmetry when we impose the constraint. Again this is similar to type-IIB supergravity: (Fermionic extension of) the action (2.1) does not have local supersymmetry, but under the self-duality constraint the equations of motion do.

3. Feynman rules and tree level 4-point amplitudes

One of the advantages of having an action, even though it must be supplemented with the constraint, is that we can construct Feynman rules for computing perturbative amplitudes. This is somewhat similar to what has been done for self-dual fields in [7].

First let us expand S_F

$$S_F = -\frac{1}{2g^2}\left\langle\!\!\left\langle (Q_B\Xi)(\eta_0\Psi) - \{(Q_B\Xi)(\eta_0\Psi) + (\eta_0\Psi)(Q_B\Xi)\}\Phi - \right.\right.$$
$$\left.\left. - \frac{1}{2}\{(Q_B\Xi)(\eta_0\Psi) - (\eta_0\Psi)(Q_B\Xi)\}\Phi\Phi + (Q_B\Xi)\Phi(\eta_0\Psi)\Phi + \cdots\right\rangle\!\!\right\rangle \tag{3.1}$$

$$= -\frac{1}{2g^2}\sum_{n\geq 0, m\geq 0}\frac{(-1)^m}{n!m!}\langle\langle (Q_B\Xi)\Phi^n(\eta_0\Psi)\Phi^m\rangle\rangle \,. \tag{3.2}$$

From cubic and higher terms of this expansion we can read off interaction vertices. Since the "anti self-dual" part should decouple, we project out the component which does not satisfy the linearized constraint $Q_B\Xi = \eta_0\Psi$. i.e. $Q_B\Xi$ and $\eta_0\Psi$ in these vertices should be replaced by $\omega \equiv \frac{1}{2}(Q_B\Xi + \eta_0\Psi)$. Then we can see that only those with odd Φs survive:

For even N,

$$\begin{aligned} -\frac{1}{2g^2} \sum_{n\geq 0, m\geq 0, n+m=N} \frac{(-1)^m}{n!m!} \langle\langle \omega\Phi^n\omega\Phi^m\rangle\rangle &= \\ = -\frac{1}{4g^2} \sum_{n\geq 0, m\geq 0, n+m=N} \frac{1}{n!m!}((-1)^m - (-1)^n) \langle\langle \omega\Phi^n\omega\Phi^m\rangle\rangle \\ = 0\,. \end{aligned} \tag{3.3}$$

For odd N,

$$\begin{aligned} -\frac{1}{2g^2} \sum_{n\geq 0, m\geq 0, n+m=N} \frac{(-1)^m}{n!m!} \langle\langle \omega\Phi^n\omega\Phi^m\rangle\rangle &= \\ = -\frac{1}{2g^2} \sum_{m>n\geq 0, n+m=N} \frac{1}{n!m!}((-1)^m - (-1)^n) \langle\langle \omega\Phi^n\omega\Phi^m\rangle\rangle \\ = -\frac{1}{g^2} \sum_{m>n\geq 0, n+m=N} \frac{(-1)^m}{n!m!} \langle\langle \omega\Phi^n\omega\Phi^m\rangle\rangle\,. \end{aligned} \tag{3.4}$$

If ω is connected to an external leg, we can safely replace it by $\eta_0\Psi$. Then we can see easily that the 3-point vertex reproduces 3-point tree level amplitudes in the first quantization formalism.

We have to give the propagators for Ξ and Ψ to complete the Feynman rules. First let us recall the propagator for Φ. [8] A convenient gauge fixing condition for the linearized gauge transformation is $G_0^-\Phi = \widetilde{G}_0^-\Phi = 0$, where $G_0^- = b_0$, $\widetilde{G}_0^- = \{Q_B, J_0^{--}\}$ and $J_0^{--} = \oint \frac{dz}{2\pi i} b(z)\xi(z)$. Under this condition the propagator $P \equiv \underbrace{\Phi\Phi}$ is given by $P = (L_0^{\mathrm{tot}})^{-2} G_0^- \widetilde{G}_0^-$. L^{tot} is the total Virasoro operator. For Ξ and Ψ, since the same gauge fixing condition cannot be imposed on the action, the kinetic term in the action cannot be used directly to compute the propagator under this gauge fixing condition, but it helps us to guess the correct form of the propagators: Only $\underbrace{\Xi\Psi}$ and $\underbrace{\Psi\Xi}$ are nonzero. Since propagating degrees of freedom should satisfy the constraint, we can think the same gauge fixing condition can be effectively imposed on Ξ and Ψ, and therefore the propagator is given by the same one as Φ: $\underbrace{\Xi\Psi} = \underbrace{\Psi\Xi} = -2P$. The factor -2 comes from the difference of the coefficients of the kinetic terms in S_B and S_F.

Strictly speaking, these propagators are given by

$$\begin{aligned} \underbrace{\Psi\Xi} &= -2\sum_i \left|i, \left(\frac{1}{2}, 0\right)\right\rangle \left\langle i, \left(-\frac{3}{2}, 2\right)\right|' (L_0^{\mathrm{tot}})^{-2} G_0^- \widetilde{G}_0^- \sum_j \left|j, \left(-\frac{1}{2}, 2\right)\right\rangle' \left\langle j, \left(-\frac{1}{2}, 0\right)\right|, \\ \underbrace{\Xi\Psi} &= -2\sum_i \left|i, \left(-\frac{1}{2}, 0\right)\right\rangle \left\langle i, \left(-\frac{1}{2}, 2\right)\right|' (L_0^{\mathrm{tot}})^{-2} G_0^- \widetilde{G}_0^- \sum_j \left|j, \left(-\frac{3}{2}, 2\right)\right\rangle' \left\langle j, \left(\frac{1}{2}, 0\right)\right|, \end{aligned} \tag{3.5}$$

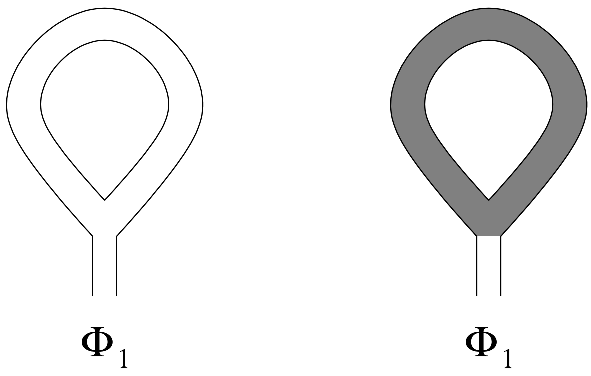

Figure 1: 1-loop tadpole diagrams.

where $\{|i,(n,m)\rangle\}$ are bases for $(n_p,n_g)=(n,m)$ states satisfying the gauge fixing condition, and $\{\langle i,(n,m)|'\}$ are conjugate bases satisfying $\langle i,(-n-1,-m+2)|'\,|j,(n,m)\rangle=\delta_{ij}$. The strips corresponding to these propagators have the Ramond boundary condition. Since we set $|i,n\rangle$ Grassmann odd, we have an additional minus sign for each fermion loop.

Practically we need the propagator between ωs:

$$\underbracket{\omega\omega}=\frac{1}{4}((Q_B\underbracket{\Xi)(\eta_0\Psi})+(\eta_0\underbracket{\Psi)(Q_B\Xi}))\,. \tag{3.6}$$

As an immediate check of these rules, let us calculate 1-loop tadpoles. As is indicated in figure 1, we have bosonic loop and fermionic loop contributions. We show fermions by shaded strips, and bosons by unshaded strips. Noting that the bosonic 3-point vertex is given by $-\frac{1}{6}\,\langle\langle\Phi\{(Q_B\Phi)(\eta_0\Phi)+(\eta_0\Phi)(Q_B\Phi)\}\rangle\rangle$, the contribution of the bosonic loop is

$$\left(-\frac{1}{6}\right)\cdot 3\cdot\left\langle\left\langle\{((Q_B\underbracket{\Phi)(\eta_0\Phi})+(\eta_0\underbracket{\Phi)(Q_B\Phi})\}\Phi_1\right\rangle\right\rangle\,. \tag{3.7}$$

The contribution of the fermion loop is

$$(-1)\cdot\frac{1}{4}\cdot\left\langle\left\langle\{(Q_B\underbracket{\Psi)(\eta_0\Xi})+(\eta_0\underbracket{\Xi)(Q_B\Psi})\}\Phi_1\right\rangle\right\rangle. \tag{3.8}$$

By $\underbracket{\Xi\Psi}=\underbracket{\Psi\Xi}=-2\,\underbracket{\Phi\Phi}$, we can see the above two contributions are in the same form. The differences are the signs and the boundary conditions on the boundary of the loops. This coincides with the expected result.

Next let us calculate on-shell 4-point tree level amplitudes with fermions. In [8] it has been shown that the bosonic part of the superstring field theory reproduces tree level on-shell 4-boson amplitude in the first quantization formalism. Therefore we expect that our Feynman rules reproduce fermion amplitudes. Since we take external legs on-shell, they satisfy the linearized equations of motion $Q_B\eta_0\Phi=Q_B\eta_0\Psi=0$. In the following we mostly follow the notation of [8], and we do not explain each step of the calculation in detail here, because much of the details is parallel to the argument in [8].

First we calculate the 4-fermion amplitude A_{FFFF}. We sum up those with 4 external fermion legs in the order of $\Psi_4\Psi_1\Psi_2\Psi_3$ and its cyclic permutations, and compare with the corresponding one in the first quantization formalism. Those in other orders can

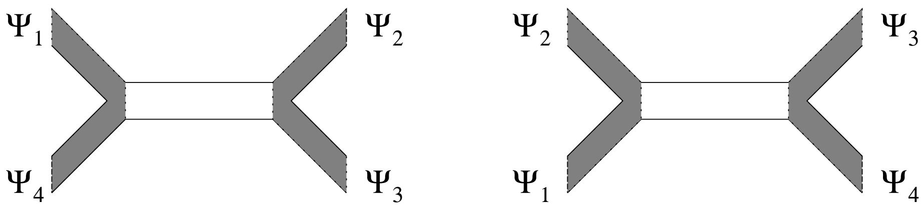

Figure 2: 4-fermion interaction.

be considered similarly. Since we have no 4-point vertex with fermions, our task is to compute "s-channel" contribution A^s_{FFFF} and "t-channel" contribution A^t_{FFFF} indicated in figure 2. In the 4-boson case the 4-boson vertex played a crucial role when we combine s- and t-channel contributions into one single integral. In the present case we expect that $A^s_{FFFF} + A^t_{FFFF}$ itself is expressed by one single integral. Let us see if this is the case.

The s-channel contribution is

$$\begin{aligned} A^s_{FFFF} &= g^{-2}\,\langle\langle(\eta_0\Psi_4)(\eta_0\Psi_1)\Phi\rangle\rangle\langle\langle\Phi(\eta_0\Psi_2)(\eta_0\Psi_3)\rangle\rangle \\ &= g^{-2}\,\langle(\eta_0\Psi_4)(\eta_0\Psi_1)P(\eta_0\Psi_2)(\eta_0\Psi_3)\rangle_W\,. \end{aligned} \tag{3.9}$$

Then we deform off the contour of

$$Q_B = \oint \frac{dz}{2\pi i} j_B(z)$$

in

$$P = (L_0^{\mathrm{tot}})^{-2} G_0^- \widetilde{G}_0^- = (L_0^{\mathrm{tot}})^{-2} G_0^- \{Q_B, J_0^{--}\}$$

away from $J^{--}(z)$, effectively replacing P by $(L_0^{\mathrm{tot}})^{-1} J_0^{--}$:

$$\begin{aligned} A^s_{FFFF} &= g^{-2}\,\big\langle(\eta_0\Psi_4)(\eta_0\Psi_1)(L_0^{\mathrm{tot}})^{-1}J_0^{--}(\eta_0\Psi_2)(\eta_0\Psi_3)\big\rangle_W \\ &= g^{-2}\int_0^\infty d\tau\,\left\langle\int_c \frac{dw}{2\pi i} J^{--}(w)(\eta_0\Psi_4)(\eta_0\Psi_1)(\eta_0\Psi_2)(\eta_0\Psi_3)\right\rangle_W \\ &= -g^{-2}\int_0^\delta d\alpha\frac{d\tau}{d\alpha}\left\langle\int_{\bar c}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)\eta_0\Psi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})\right\rangle. \end{aligned} \tag{3.10}$$

Readers can find the definitions of τ, α, δ, c, $\bar c$, $w = w(z)$ and W in [8].

Similarly,

$$\begin{aligned} A^t_{FFFF} &= g^{-2}\,\langle\langle(\eta_0\Psi_3)(\eta_0\Psi_4)\Phi\rangle\rangle\langle\langle\Phi(\eta_0\Psi_1)(\eta_0\Psi_2)\rangle\rangle \\ &= g^{-2}\int_\delta^1 d\alpha\frac{d\tau}{d\alpha}\left\langle\int_{\bar c}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)\eta_0\Psi_3(\alpha^{-1})\eta_0\Psi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)\right\rangle. \end{aligned} \tag{3.11}$$

We see that the sum of A^s_{FFFF} and A^t_{FFFF} is one single integral over the moduli space, and coincides with the amplitude in the first quantization formalism:

$$\begin{aligned} A_{FFFF} &= -g^{-2}\int_0^1 d\alpha\frac{d\tau}{d\alpha}\left\langle\int_{\bar c}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)\eta_0\Psi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})\right\rangle \\ &= -g^{-2}\int_0^1 d\alpha\left\langle\int d^2z\,\mu_\alpha(z,\bar z)J^{--}(z)\eta_0\Psi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})\right\rangle, \end{aligned} \tag{3.12}$$

where $\mu_\alpha(z,\bar z)$ is the Beltrami differential corresponding to the α modulus.

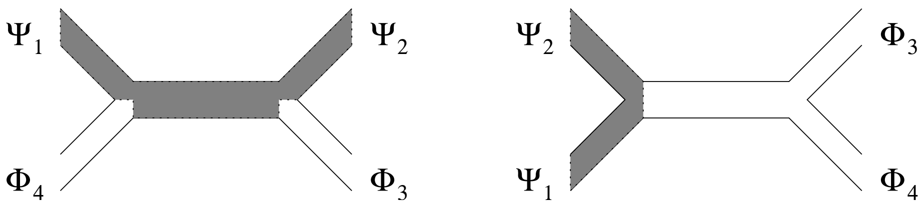

Figure 3: 2-boson 2-fermion interaction in the order of $\Phi_4\Psi_1\Psi_2\Phi_3$.

The second example is the 2-boson 2-fermion amplitude A_{FFBB} in the order of $\Phi_4\Psi_1\Psi_2\Phi_3$, indicated in figure 3. The "s-channel" contribution is

$$\begin{aligned}
A^s_{FFBB} &= g^{-2}\,\langle\langle\Phi_4(\eta_0\Psi_1)\omega\rangle\rangle\langle\langle\omega(\eta_0\Psi_2)\Phi_3\rangle\rangle \\
&= -\frac{1}{2}g^{-2}\Big[\langle(Q_B\Phi_4)(\eta_0\Psi_1)P(\eta_0\Psi_2)(\eta_0\Phi_3)\rangle_W + \langle(\eta_0\Phi_4)(\eta_0\Psi_1)P(\eta_0\Psi_2)(Q_B\Phi_3)\rangle_W\Big] \\
&= \frac{1}{2}g^{-2}\int_0^\delta d\alpha\frac{d\tau}{d\alpha}\bigg\langle\int_{\bar{c}}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)\Big[Q_B\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)\eta_0\Phi_3(\alpha^{-1}) + \\
&\qquad + \eta_0\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)Q_B\Phi_3(\alpha^{-1})\Big]\bigg\rangle. \qquad (3.13)
\end{aligned}$$

In the second line we deformed off contours of $Q_B = \oint \frac{dz}{2\pi i}j_B(z)$ and $\eta_0 = \oint \frac{dz}{2\pi i}\eta(z)$ away from Ψ and Ξ in the propagators to other fields.

Similarly the "t-channel" contribution is

$$\begin{aligned}
A^t_{FFBB} &= 3\cdot\left(-\frac{1}{6}\right)g^{-2}\,\langle\langle(\eta_0\Psi_1)(\eta_0\Psi_2)\Phi\rangle\rangle\langle\langle\Phi\{(Q_B\Phi_3)(\eta_0\Phi_4)+(\eta_0\Phi_3)(Q_B\Phi_4)\}\rangle\rangle \\
&= -\frac{1}{2}g^{-2}\,\langle(\eta_0\Psi_1)(\eta_0\Psi_2)P\{(Q_B\Phi_3)(\eta_0\Phi_4)+(\eta_0\Phi_3)(Q_B\Phi_4)\}\rangle_W \\
&= \frac{1}{2}g^{-2}\int_\delta^1 d\alpha\frac{d\tau}{d\alpha}\bigg\langle\int_{\bar{c}}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)\Big[Q_B\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)\eta_0\Phi_3(\alpha^{-1}) + \\
&\qquad + \eta_0\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)Q_B\Phi_3(\alpha^{-1})\Big]\bigg\rangle. \qquad (3.14)
\end{aligned}$$

Again the sum of these contributions is expressed by one single integral and gives the amplitude of the first quantization formalism:

$$\begin{aligned}
A_{FFBB} &= \frac{1}{2}g^{-2}\int_0^1 d\alpha\frac{d\tau}{d\alpha}\bigg\langle\int_{\bar{c}}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)\Big[Q_B\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)\eta_0\Phi_3(\alpha^{-1}) + \\
&\qquad + \eta_0\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)Q_B\Phi_3(\alpha^{-1})\Big]\bigg\rangle \qquad (3.15) \\
&= g^{-2}\int_0^1 d\alpha\frac{d\tau}{d\alpha}\bigg\langle\int_{\bar{c}}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)Q_B\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)\eta_0\Phi_3(\alpha^{-1})\bigg\rangle \\
&= g^{-2}\int_0^1 d\alpha\bigg\langle\int d^2z\,\mu_\alpha(z,\bar{z})J^{--}(z)Q_B\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Psi_2(\alpha)\eta_0\Phi_3(\alpha^{-1})\bigg\rangle.
\end{aligned}$$

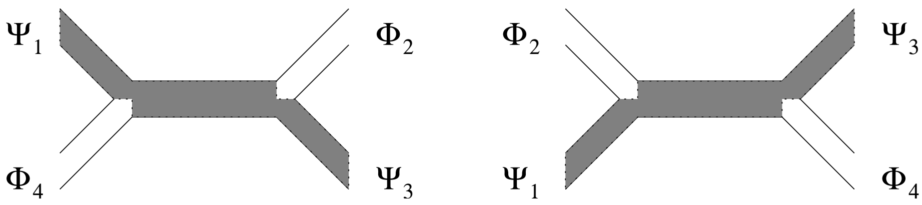

Figure 4: 2-boson 2-fermion interaction in the order of $\Psi_1\Phi_2\Psi_3\Phi_4$.

In the second equality we exchanged Q_B and η_0 on Φ_3 and Φ_4 in the second term of the first equality by contour deformation. This manipulation leaves a total derivative term with respect to α, but by the canceled propagator argument we can drop it.

Finally we compute the 2-boson 2-fermion amplitude A_{FBFB} in the order of $\Psi_1\Phi_2\Psi_3\Phi_4$ indicated in figure 4.

$$\begin{aligned}
A^s_{FBFB} &= -g^{-2}\,\langle\langle\Phi_4(\eta_0\Psi_1)\omega\rangle\rangle\langle\langle\omega\Phi_2(\eta_0\Psi_3)\rangle\rangle \\
&= \frac{1}{2}g^{-2}[\langle(Q_B\Phi_4)(\eta_0\Psi_1)P(\eta_0\Phi_2)(\eta_0\Psi_3)\rangle_W + \langle(\eta_0\Phi_4)(\eta_0\Psi_1)P(Q_B\Psi_2)(\eta_0\Psi_3)\rangle_W] \\
&= -\frac{1}{2}g^{-2}\int_0^\delta d\alpha\frac{d\tau}{d\alpha}\Bigg\langle\int_{\bar c}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)\Big[Q_B\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Phi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})+ \\
&\qquad + \eta_0\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)Q_B\Phi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})\Big]\Bigg\rangle. \qquad (3.16)
\end{aligned}$$

$$\begin{aligned}
A^t_{FBFB} &= -g^{-2}\,\langle\langle(\eta_0\Psi_1)\Phi_2\omega\rangle\rangle\langle\langle\omega(\eta_0\Psi_3)\Phi_4\rangle\rangle \\
&= \frac{1}{2}g^{-2}[\langle(\eta_0\Psi_1)(\eta_0\Phi_2)P(\eta_0\Psi_3)(Q_B\Phi_4)\rangle + \langle(\eta_0\Psi_1)(Q_B\Phi_2)P(\eta_0\Psi_3)(\eta_0\Phi_4)\rangle] \\
&= -\frac{1}{2}g^{-2}\int_\delta^1 d\alpha\frac{d\tau}{d\alpha}\Bigg\langle\int_{\bar c}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)\Big[Q_B\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Phi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})+ \\
&\qquad + \eta_0\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)Q_B\Phi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})\Big]\Bigg\rangle. \qquad (3.17)
\end{aligned}$$

Again we reproduce the amplitude in the first quantization formalism:

$$\begin{aligned}
A_{FBFB} &= -\frac{1}{2}g^{-2}\int_0^1 d\alpha\frac{d\tau}{d\alpha}\Bigg\langle\int_{\bar c}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)\Big[Q_B\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Phi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})+ \\
&\qquad + \eta_0\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)Q_B\Phi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})\Big]\Bigg\rangle \qquad (3.18) \\
&= -g^{-2}\int_0^1 d\alpha\frac{d\tau}{d\alpha}\Bigg\langle\int_{\bar c}\frac{dz}{2\pi i}\frac{dz}{dw}J^{--}(z)Q_B\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Phi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})\Bigg\rangle \\
&= -g^{-2}\int_0^1 d\alpha\Bigg\langle\int d^2z\mu_\alpha(z,\bar z)J^{--}(z)Q_B\Phi_4(-\alpha^{-1})\eta_0\Psi_1(-\alpha)\eta_0\Phi_2(\alpha)\eta_0\Psi_3(\alpha^{-1})\Bigg\rangle.
\end{aligned}$$

Thus all types of 4-point tree level amplitude coincide with those of the first quantization formalism.

4. Discussion

Led by the analogy to type-IIB supergravity, we have given a covariant action for the fermion field with a constraint, and construct Feynman rules for it. We have calculated on-shell tree level 4-point amplitudes with fermions and have seen that they coincide with those of the first quantization formalism.

In our calculation of amplitudes we used only 3-point and 4-point vertices. (In fact the 4-point vertices with fermions are absent. Therefore our calculation shows the correctness of their absence.) To confirm the correctness of higher vertices, we have to compute higher correlators.

To compute loop amplitudes we need fermions, even if all the external legs are bosons, because fermions circulate through loops. Now that we have Feynman rules for fermions, in principle we can calculate any loop amplitude. It is very interesting to see whether any cancellation expected from supersymmetry occurs among loop amplitudes. Another interesting issue is to compute anomalies which come from fermion loops.

Pursuing the analogy to field theories with self-dual fields further, it is natural to consider PST formalism-like formulation [4] i.e. introducing auxiliary fields and constructing an action without any constraint. It is intriguing to see if such formulation is possible in the open superstring field theory.

Acknowledgments

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References

[1] N. Berkovits, *SuperPoincaré invariant superstring field theory*, *Nucl. Phys.* **B 450** (1995) 90 [hep-th/9503099].

[2] N. Berkovits, *The ramond sector of open superstring field theory*, *J. High Energy Phys.* **11** (2001) 047 [hep-th/0109100].

[3] N. Marcus and J.H. Schwarz, *Field theories that have no manifestly Lorentz invariant formulation*, *Phys. Lett.* **B 115** (1982) 111.

[4] P. Pasti, D.P. Sorokin and M. Tonin, *On Lorentz invariant actions for chiral p-forms*, *Phys. Rev.* **D 55** (1997) 6292 [hep-th/9611100];
G. Dall'Agata, K. Lechner and M. Tonin, *$D = 10$, $N = IIB$ supergravity: Lorentz-invariant actions and duality*, *J. High Energy Phys.* **07** (1998) 017 [hep-th/9806140].

[5] N. Berkovits, A. Sen and B. Zwiebach, *Tachyon condensation in superstring field theory*, *Nucl. Phys.* **B 587** (2000) 147 [hep-th/0002211].

[6] C. Wendt, *Scattering amplitudes and contact interactions in Witten's superstring field theory*, *Nucl. Phys.* **B 314** (1989) 209.

[7] L. Alvarez-Gaumé and E. Witten, *Gravitational anomalies*, *Nucl. Phys.* **B 234** (1984) 269.

[8] N. Berkovits and C.T. Echevarria, *Four-point amplitude from open superstring field theory*, *Phys. Lett.* **B 478** (2000) 343 [hep-th/9912120].